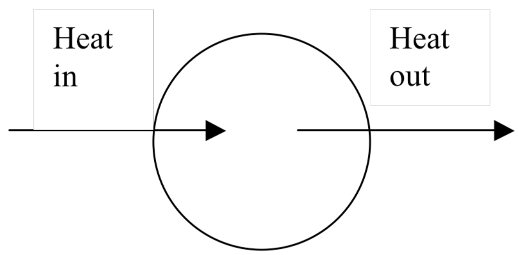
Transfer Function Development
The Thermometer[1]

Assumptions:

Lumped	- All thermal resistance resides in fluid film surrounding bulb
Parameter	- All thermal capacity is in the mercury and it is uniform throughout
Model	- The glass does not expand
	- Thermometer is initially at steady state



Flows Sum

$$hA(x - y) - mC\frac{dy}{dt} - 0 = 0$$

where:
- H = film coefficient of heat transfer [=] $\frac{BTU}{hr \bullet ft^2 \bullet {}^\circ F}$
- A = Area of bulb [=] ft^2
- x = Fluid Temp [=] °F
- y = Hg Temp [=]°F
- C = Heat capacity of Hg [=] $\frac{BTU}{lb_m {}^\circ F}$
- m = Mass of mercury [=] lb_m
- t = time [=] hr

It is not convenient to work in terms of absolute temperature, instead solve the problem in terms of deviations

For $t \leq 0$ Thermometer is at steady state

So

$$hA(x_s - y_s) = 0$$
$$x_s = y_s$$

So

$$h \bullet A \bullet \left[(x - x_s) - (y - y_s)\right] = m \bullet C \bullet \frac{d(y - y_s)}{dt}$$

$$\frac{d(y - y_s)}{dt} = \frac{dy}{dt}$$ since y_s is a constant

If

$X = x - x_s$
$Y = y - y_s$

So

$$X - Y = \frac{m \bullet C}{h \bullet A} \bullet \frac{dY}{dt}$$

Letting

$$\frac{m \bullet C}{h \bullet A} = \tau$$

$$X - Y = \tau \bullet \frac{dY}{dt}$$

Taking the LaPlace Transform

$$\mathcal{L}\{X(t)\} - \mathcal{L}\{Y(t)\} = \tau \bullet s \bullet \mathcal{L}\{Y(t)\}$$

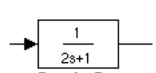
Or

$$\frac{\mathcal{L}\{Y(t)\}}{\mathcal{L}\{X(t)\}} = \frac{1}{\tau \bullet s + 1} = \frac{Y(s)}{X(s)}$$

Where τ = Time Constant [=] hours

This is a:
- First Order System
- First Order Lag
- Single Exponential Stage

$$\frac{Y(s)}{X(s)} = \text{Transfer Function} = G(s)$$



This is the most commonly used form of a sensor

Transfer functions represent a very useful form for describing a system. Since they are a description that does not include either the input or the output, they represent a general method for capturing the behavior of the system.

In the case of the above problem, X(s) is our input and Y(s) our output. This allows us to write:

$$Y(s) = X(s) \bullet G(s),$$

implying that for any input we can compute the system response by multiplying the transfer function, G(s), by the input.

Step Response

To examine the response of the thermometer to a step input in temperature, we would note that

$$X(s) = \frac{A}{s}$$

where: A = the magnitude of the step change in temperature [=] °F

Then:

$$Y(s) = \frac{A}{s} \bullet \frac{1}{\tau + 1}$$

we can rearrange this equation into the format

$$Y(s) = \frac{A/\tau}{s \bullet \left(s + 1/\tau\right)}$$

The solution requires that partial fractions are used, so:

$$= \frac{C_1}{s} + \frac{C_2}{s + 1/\tau}$$

Where: $C_1 = A$
$C_2 = -A$

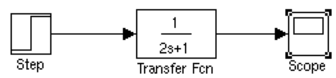
The solution is

$$Y(t) = 0 \quad \text{for} \quad t \leq 0$$

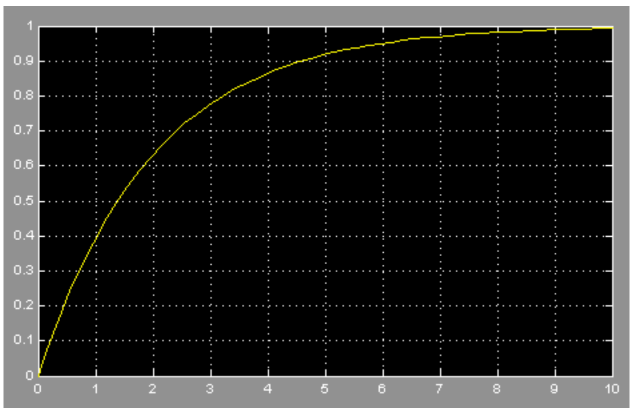
$$Y(t) = A \bullet \left(1 - e^{-t/\tau}\right) \text{ for } t \geq 0$$

Note that this solution has two components. The "A•1" portion represents the steady-state response that occurs after long time periods. The "$A \bullet -e^{-t/\tau}$" portion represents the transient response that dominates right after the input is applied.

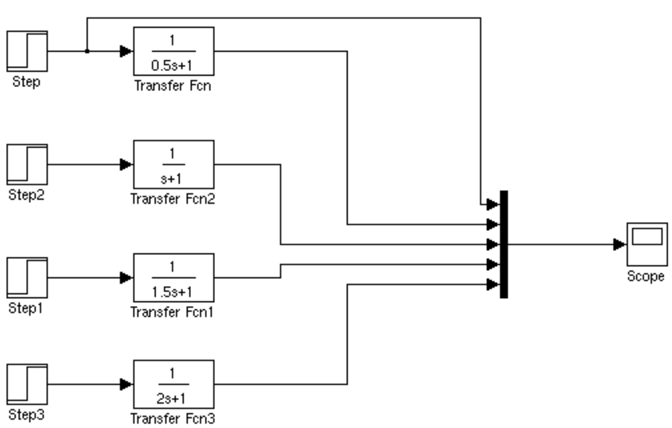
When this system is created in Simulink, the resulting model appears below:



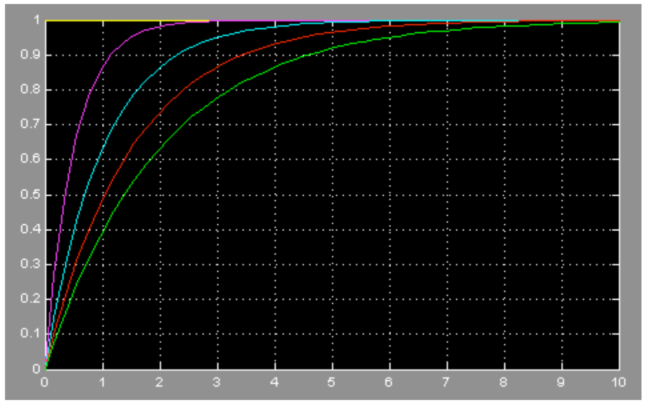
In this model, $\tau = 2$. The result from running the model is:



This model only considers a single value for τ. However, is would be useful to compare the effect of different values for τ in the same scope. This can be accomplished by building multiple versions of the model, varying τ. The outputs from the models are input into a single scope using the "Mux" block. In the model below, I have also included the input into the transfer function on the scope for comparison.



Running this version of the model yields:



The bottom (green) curve shows the response for **τ** = 2, while the uppermost curve (purple) shows the response for **τ** = 0.5. Horizontal (yellow) line at y = 1 is the input unit step function.

Some terms associated with τ and A systems response to a step change

1. The time constant and a definition of steady-state

Elapsed Time		% A	
1τ	⇒	62.3	
2τ	⇒	86.5	
3τ	⇒	95.0	
4τ	⇒	98.0	Usually defined as Steady-State
5τ	⇒	99.0	

2. System DC gain = steady–state gain for the case where the output has a final value

3. At $\frac{t}{\tau} = 0 \qquad \frac{dY}{dt} = 1$

4. The response of any step change may be determined from the response for "A" by multiplying the result for "A" by $\frac{desired}{A}$

Sinusoidal Input

This input is represented by the equations

$$X = 0 \qquad \text{for } t < 0$$

And

$$X = A \bullet \sin(\varpi \bullet \tau) \qquad \text{for } t \geq 0$$

Where: A = amplitude
ϖ = frequency [=] radians

From the table, the Laplace transform of this equation is

$$X(s) = \frac{A \bullet \varpi}{s^2 + \varpi^2}$$

Recalling that the transfer function approach allows us to state

$$Y(s) = X(s) \bullet G(s)$$

And, for the thermometer our transfer function is

$$G(s) = \frac{1/\tau}{s + 1/\tau}$$

so

$$Y(s) = \frac{A \bullet \varpi}{s^2 + \varpi^2} \bullet \frac{1/\tau}{s + 1/\tau}$$

Solving this equation using partial fractions yields

$$Y(t) = \left(\frac{A\varpi}{\tau^2 \bullet \varpi^2 + 1}\right) e^{-t/\tau} - \left(\frac{A \bullet \varpi \bullet \tau}{\tau^2 \bullet \varpi^2 + 1}\right) \cos(\varpi \bullet t) + \left(\frac{A}{\tau^2 \bullet \varpi^2 + 1}\right) \sin(\varpi \bullet t)$$

Noting the trigonometric identity

$$p \bullet \cos A + q \bullet \sin A = r \bullet \sin(A + \theta)$$

where

$$r = \sqrt{p^2 + q^2}$$

$$\tan\theta = \frac{p}{q}$$

we have

$$Y(t) = \left(\frac{A\varpi}{\tau^2 \bullet \varpi^2 + 1}\right)e^{-t/\tau} + \left(\frac{A}{\sqrt{\tau^2 \bullet \varpi^2 + 1}}\right)\sin(\varpi \bullet t + \phi)$$

where $\phi = \tan^{-1}(-\varpi \bullet \tau)$

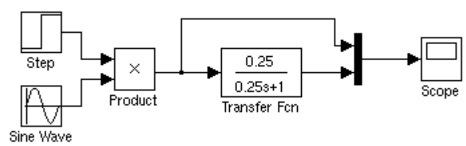
This equation has two components, a transient component that goes to 0

$$\left(\frac{A\varpi}{\tau^2 \bullet \varpi^2 + 1}\right)e^{-t/\tau}$$

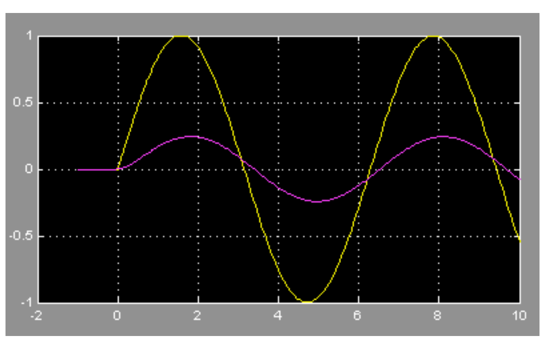
and a steady-state component

$$\left(\frac{A}{\sqrt{\tau^2 \bullet \varpi^2 + 1}}\right)\sin(\varpi \bullet t + \phi)$$

Building this model in Simulink yields



Note that, in order for the input to remain at 0 for $t \le 0$, the sinusoidal input has been multiplied by a unit step function occurring at t = 0. In this model, $\tau = 4$, $\varpi = 1$ Hz, and A = 1. The output from the system is:



There are a couple of important factors to note regarding the *steady-state* output of the equation. They are:

1. The output frequency of the steady-state frequency is the same as that of the input.

2. The ratio of the output amplitude to the input amplitude is:

$$\frac{1}{\sqrt{\tau^2 \bullet \varpi^2 + 1}}$$

This is always less than 1. Thus, the amplitude of the output is *attenuated.*

3. The attenuation of the input increases as the frequency, ϖ, increases. Thus, processes with the format

$$G(s) = \frac{1/\tau}{s + 1/\tau} \quad \text{or} \quad G(s) = \frac{1}{\tau \bullet s + 1}$$

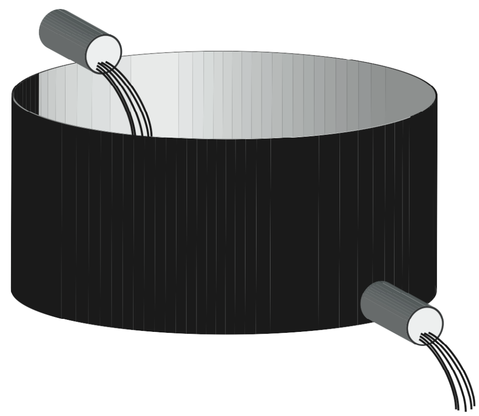
act as low pass filters

4. The output lags behind the input by an angle $|\ \phi\ |$. Since ϕ is always negative, *phase lag* always occurs. ($\phi > 0$ is phase lead) The phase lag can never exceed $\pi/2$, but approaches it asymptotically

Transfer Function Development
A Surge Tank

Surge tanks are used to smooth out an irregular flow. They are used in a variety of systems. For example, a basin might be designed to catch the flow of water off of a parking lot after a storm. Since parking lots are impervious to rain, even a relatively small storm generates a large volume of water capable of overflowing a drainage ditch. The basin contains that volume of water releasing it at a controlled rate.

Another example might be a reservoir designed to regulate the flow of sewage into a waste treatment plant. Since the generation of waste is irregular, the extreme example being the infamous Superbowl flush that occurs at halftime, the surge tank allows the plant to be designed for a flow rate that is closer to the average than it is to the maximum.



Assumptions:
- The desired output for the system is the head required to handle the flow into the tank
- The tank has a uniform cross-section
- The output from the tank has a constriction (valve) limiting its flow
- The resistance to flow generates a linear relationship between head and flow

Then:

q_0 = the volumetric flow rate [=] volume/time
R = the resistance of the constriction
h = the head created in the tank
A = the cross-sectional area of the tank

We can write:

$$q_0 = \frac{h}{R}$$

Analysis of this system is based on a mass balance where

Mass Flow In – Mass Flow Out = Rate of Mass Accumulation in the Tank

To relate the mass balance to the flow rate, we use the density of the fluid, ρ. The density is assumed to be constant. Then the mass balance becomes:

$$\rho \bullet q(t) - \rho \bullet q_0(t) = \frac{d(\rho \bullet A \bullet h)}{dt}$$

Dividing by ρ yields

$$q(t) - q_0(t) = A\frac{d(h)}{dt}$$

Since we are trying to relate the flow in, q(t), to the head, h, we will convert q_0 to head, yielding:

$$q - \frac{h}{R} = A \bullet \frac{dh}{dt}$$

As with the previous problem, we will be working with the process in terms of deviation from steady-state. Therefore, we assume that the tank is at steady-state prior t = 0. At steady-state

$$\frac{dh_s}{dt} = 0$$

so

$$q_s - \frac{h_s}{R} = 0$$

To convert the analysis to deviation term, we subtract the steady-state version of the equation from the general form of the equation, yielding

$$(q - q_s) = \frac{(h - h_s)}{R} + A \bullet \frac{d(h - h_s)}{dt}$$

Next, we define our deviation variables

$$Q = q - q_s$$

$$H = h - h_s$$

And note that at steady-state for $t < 0$,

$$H(0) = 0$$

And rewrite our differential equation to

$$Q(s) = \frac{1}{R} H(s) + A \bullet s \bullet H(s)$$

We can rearrange this equation into the standard transfer function format

$$G(s) = \frac{H(s)}{Q(s)}$$

where

$$\frac{H(s)}{Q(s)} = \frac{R}{\tau \bullet s + 1}$$

where $\tau = A \bullet R$

This transfer function is essentially the same as that found for the thermometer, only the variables have been changed.

Impulse Input

Problem Statement

A water retention pond is associated with the parking lot of a football stadium. The pond has dimensions of 10 m x 5 m and is of uniform cross section. A stream with a flow rate of 1 m^3/min continuously flows through the pond, keeping the pond partially full. The drain pipe exiting the pond has a resistance of 1/20 m/(m^3/min). At t = 0, a cloud burst lasting a minute occurs, dumping an additional 40 m^3 of water onto the pond and the parking lot.

From this description, we can generate a transfer function

$$\frac{H(s)}{Q(s)} = \frac{R}{\tau \bullet s + 1}$$

Where:
$R = 1/20$ [=] m/(m^3/min)
$A = 5 \bullet 10 = 50$ [=] m^2
$\tau = A \bullet R = 2.5$ [=] 1/min

so

$$\frac{H(s)}{Q(s)} = \frac{0.05}{2.5 \bullet s + 1}$$

Because a large volume of water is added to the pond over a short period of time, we can treat the addition as an impulse function with a magnitude of 40. In this case,

$$Q(s) = 40 \bullet 1$$

So

$$H(s) = 40 \bullet \frac{0.05}{2.5 \bullet s + 1}$$

Rearranging into the format of the Laplace transforms table

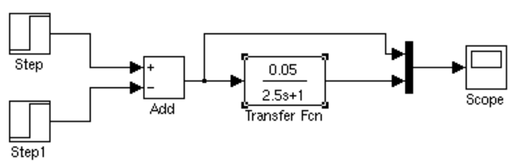
$$H(s) = 0.8 \bullet \frac{1}{s + 0.4}$$

Taking the inverse Laplace transform yields

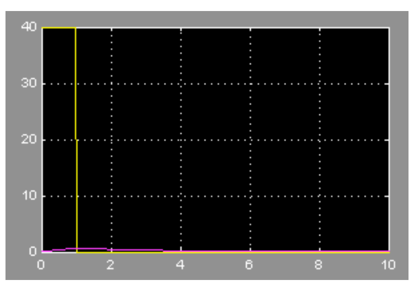
$$H(t) = 0.8 \bullet e^{-2.5 \bullet t}$$

Note that the exponential decay occurs from a maximum value of 0.8 meters of additional height of the water level.

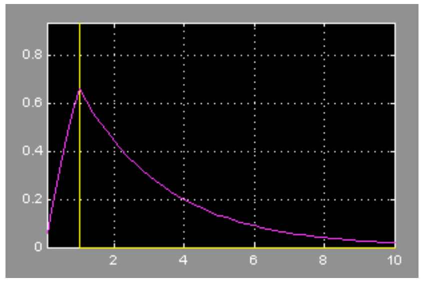
While the we approximated the cloudburst as an impulse function to simplify the analytical solution, This simplification is not beneficial when we build the model in Simulink. We can model the system as shown below:



The pulse of rain entering the pond is created by inputting a step function of height 40 and one minute later subtracting another step function of height 40. The output from the scope is



The yellow line is the input pulse and the pink line is the additional height of the pond. If we enlarge this plot, clipping the input pulse, we have



References

[1] D. R. Coughanowr and L. B. Koppel, *Process Systems Analysis and Control*. New York: Mc-Graw Hill Book Company, 1965.